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THOUGHT VALUES IN BEGINNING ALGEBRA.

NOT long since an algebra teacher, whose pupils had studied the subject for a half year before she took them, remarked to her principal that they did not seem to know their previous work very well. He replied: "I don't see why that should be so. Mr. A., who had them in that work, is certainly a good drillmaster, and that is the important thing in beginning algebra." In my own senior review classes, in which the work was almost altogether at first a discussion of definitions and principles, some pupils were sure to ask after the first few weeks: "When are we going to begin the real algebra? This so far is just the introduction, isn't it?" I always said, "What do you mean by 'real algebra'?" and the answer was invariably, "Why, the working of examples," and sometimes "examples having letters in them instead of figures" was added.

These two incidents, one from the teacher's and one from the pupil's standpoint, illustrate what is perhaps the fundamental failure in present algebra teaching. The average teacher seems to regard ability to solve problems as the end in algebra work, and drill as the means to this end, and consequently most boys and girls who have had algebra think that it means only drill and grind. This is certainly not as it should be; it is the principle, not the problem, that is the heart and soul of mathematics, and knowledge of principles, together with the ability to think

abstractly, that results from the right teaching of principles, should be the immediate aim of the algebra teacher. How is this aim to be realized? First of all, not by assigning a page or so of definitions for the first lesson. One may safely say that elementary work should never begin with definitions. Of course some teachers will urge that it means a great loss of time if the definitions are not mastered to start with. Those who advocate the plan discussed later in this paper grant this freely and completely; they are quite willing that it should be so; time is lost, but power is gained, and that is an exchange which the teacher should always be willing to make. The desired end is to be reached, first, by following some simple, consistent, and well-arranged system of principles; second, by teaching these principles according to the so-called "inductive" plan. The theory of this plan is too well understood to need explanation here; but there are certain requisites, not always kept in mind, without which it cannot possibly succeed. Some of the most important of these are as follows:

1. A good understanding between teacher and pupils. If this exists, and the pupils are prepared for the work and so interested in it, matters of discipline and order will usually take care of themselves.

2. Both material and method must be fit; that is, not only logical from the teacher's standpoint, but adapted to the experience of the pupil. These do not necessarily conflict. When pupils have reached the logical stage in their development the logical method is also the psychological one.

3. Time—plenty of it. It is thought-work, and thinking cannot be hurried or done to order.

4. Notebooks. Much so-called inductive work is mere talk, resulting in nothing of permanent value. Results should be obtained. Principles and definitions should be thought out, formulated, written in notebooks, and learned (not memorized—there is a difference). In other words, the pupil should not only have the idea, but he should also have an adequate statement of it at his command.

5. "Induction" should always be followed by "deduction."

Principles should not only be discovered, formulated, and learned, but they should be immediately applied.

6. In the class there must be both "team work" and "individual playing." "Team work" means that the class all think together; it will not usually mean that they all talk together, though an occasional reply in concert may be worth while if it is spontaneous. It must not mean that the class all talk at once, each one saying a different thing. "Individual playing" means that one person acts as spokesman for the class. It may mean that he is the only one in the class who has thought the matter clear through and formulated it, but it must not mean that he is the only one who has thought at all.

7. The preparatory work outside of class should be in general thoroughly and conscientiously individual.

8. Occasional oral and written tests are necessary to enable both teacher and pupils to see whether results are being obtained or not. They also teach pupils to think coherently in a crisis, and, if of the right sort, they furnish valuable reviews.

Because it seems impossible to make plain the principles of the method under discussion without conrete illustrations, the following stenographic reports of two beginning algebra lessons are submitted. The reports can convey, of course, only the matter of the work; the spirit of it must be left to the imagination of the reader. Those who have had the experience of seeing in cold and inflexible type thoughts of their own that were vital and glowing when they were spoken will surely be charitable in their judgments.

The class in question numbers twenty-six and is of average age and intelligence. So far this year the work in mathematics has been theoretical work (definitions, principles, and demonstrations) in arithmetic. The lessons reported were not "prepared;" they were given at 9 o'clock in the morning and lasted, the first one hour, and the second forty-five minutes. At the first lesson there were twenty visitors present; the second was given under normal conditions. *T.* in the reports is for the teacher and *A.* for answer. No attempt is made to give actual names, or any names at all except in special cases. The reports

do not show, either, which questions were asked of the class as a whole and which of individual pupils. Every member of the class was called upon at least once during each recitation.

LESSON I., DECEMBER 4, 1901.

T. Our lesson this morning is simply a lesson in thinking; you will not need any notebooks for a time at least.

(Without further suggestion pupils cleared their desks. Slight pause.)

T. (writing 5 upon the board). Does this 5 mean anything to you? (There was no response, either by raising hands or orally). *T.* (after a pause). I am glad to see that my question meets with no response. It is just as it should be. Now that you have thought it over, how many are ready to answer. (A good many hands were raised). Well, Louis?

Louis. It means five units.

T. What does that mean?

Louis. It means five ones.

T. What does that mean finally?

Louis. Five whole numbers—five whole things.

T. The second part of your answer is right. Anyone else?

Frank. I do not think it means anything of itself; it is a sign.

Minnie. It is a symbol representing five units.

T. You have anticipated my next question, so I will pass on to another. Did this 5 always mean something to you? Does it mean anything to the youngest child in the kindergarten now?

Gladys. It doesn't mean anything to the child; at least he doesn't know the full meaning of it.

T. How did it come to have a meaning to you?

A. It was seeing it with relation to other numbers.

T. What do you mean? How did you get your own idea of five in the first place?

A. By being shown with pegs and things.

T. Are there any other answers?

A's. (1) By the counting frame. (2) Because there were five people in our family. (3) Because we have five fingers on each hand. (4) By the number of straight lines used in the figure.

T. (Writing 5 upon the board and rounding the two lower corners). That is very interesting, but we will not go into it further now. Who will give a name for this kind of five—five fingers, five people, and so on?

A. Concrete.[1]

T. To the kind I wrote first upon the board?

A. Abstract.

[1] To insist upon "complete" answers in this kind of work is fatal.

T. The first step in number is to get the concrete idea of number, the concrete idea of five, say; the next step is to go to abstract but still particular 5, 6, 7. (Writes 6, 7 after the 5); then to more general and more abstract a, b, c (writes a, b, c under the 5, 6, 7), and then to most general and most abstract x, y, z (writes). The last we shall not consider today (erases x, y, z, writes a by itself on the board). Does this a mean anything — anything that has to do with number I mean?

A's. (1) Yes; in writing outlines we used letters instead of figures to indicate some of the divisions. (2) In proportion we used letters sometimes to stand for the unknown quantity.[1]

T. Then this step will not be so difficult as it might be. Let us see — If John has two cents and James has three, how many have both?

A. Five cents.

T. Ellis, put this on the board with a sign to indicate the operation. (Ellis writes

$$\begin{array}{r} 2 \\ +3 \\ \hline 5. \end{array}$$

)[2]

T. Will some one write it in another way?

A's (1)

$$\begin{array}{l} 2\,c \\ 3\,c \\ \hline 5\,c \end{array}$$

(2) $2+3=5$.

Pupil. I think I can write it still another way.[3]

T. (rather grudgingly). You may try it.

(Pupil wrote

$$\begin{array}{l} 2\,c=1\,c+1\,c \\ 3\,c=1\,c+1\,c+1\,c \\ \hline 5\,c=1\,c+1\,c+1\,c+1\,c+1c. \end{array}$$

)[4]

T. Thank you very much. That is exactly what I had planned to take up in the next lesson, and it will save us a good deal of time. We will leave it now and return to it again today if we can. I will now rub out the result in this example (teacher rubs out the $=5$ in the $2+3=5$, leaving the $2+3$). Will someone bring out a distinction between this as it now stands and this other case? (Pointing to

$$\begin{array}{r} 2 \\ +3 \\ \hline 5. \end{array}$$

)

[1] I had expected the answer *no* to this question.

[2] If children have been taught from the beginning of their number work to put the proper sign before the lower number, it helps greatly when they get to algebra.

[3] I nearly spoiled the best thing in the lesson here by saying, "I think we have had examples enough."

[4] This was wholly unexpected, and necessitated a reconstruction of the lesson plan.

A. In the second the operation is performed; in the first it is stated.

T. Another word for "stated."

A. Indicated.

Pupil. I think it is merely stated in a different way.

T. Very well, if you are satisfied with the result $2+3$, so much the better. Let us have several examples now; please indicate the operation instead of performing it. John has six cents and James has five; how many have both?

A. $6+5$ cents. (Several such examples were given).

T. (finally). John has a cents and James has b cents; how many have both?

A. $a+b$ cents.

T. John has c cents and James has d cents; how many have both?

A. $c+d$ cents.

T. I do not see that it is necessary to go any further with this. We are now ready perhaps for two statements of differences between what we are now doing, which is usually called "algebra," but which is really only literal arithmetic, and what we have been doing under the name of "arithmetic." There is really no hard and fast line between the two. (Draws a vertical line on the board.) We cannot say that everything on one side of this line is algebra, and everything on the other side is arithmetic. They run into each other. Still there are some distinctions that are commonly made. Who will suggest one?

A. Letters may represent any amount you want them to.[1]

T. I do not understand your answer. Go a little bit further. How do you come by the letters to start with? (Pause.) We are dealing with number in both cases. What do we commonly use in arithmetic to represent number?

(A brief discussion enabled the teacher to write upon the board this statement: 1. In "arithmetic" digits (figures) are commonly used to represent number, while in "algebra" letters are so used.)

T. Another difference, Harry?

A somewhat longer cross-examination led to this result:

2. Processes performed in arithmetic are indicated in algebra.

T. These are not yet entirely right; figures as well as letters are used, and operations are sometimes performed as well as indicated in algebra. We will show this by some examples. John has a cents; and James has a cents; how many have both? Who is ready?

A. $a+a$ cents.

T. Right, but there is another way of doing it.

Pupil. *A* times a cents.

T. Let us see. If you have 2 cents and I have 3 cents both of us have two times three cents, or 6 cents. Is that right?

[1] I failed to see the point in this answer.

A. No. (Pause.)

T. We will leave this, and get at it in another way. (Writes on the board; class gives answers together without direction.)

T. (1) 2 horses.
$+3$ horses.

Class. 5 horses.

T. (2) 2 pounds.
$+3$ pounds.

Class. 5 pounds.

T. (3) $2\ c$ (for cents).
$+3\ c$ (for cents).

Class. $5\ c$ (for cents).

T. (4) $2\ a$ (for acres).
$+3\ a$ (for acres).

Class. $5\ a$ (for acres).

T. Now, this time a is not to stand for *acres*, but for *anything*.

(*T.* writes (5) $2\ a$ (for anything).
$+3\ a$ (for anything).

Class says, $5\ a$, when a stands for anything.)

T. (6) $2\ ab$.
$+3\ ab$.

Class. $5\ ab$.

T. (7) $2\ abc$.
$+3\ abc$.

Class. $5\ abc$.

T. I cannot resist the temptation to go one step further.

(Writes $2\ a^2\ b^2\ c^2$
$+\ 3\ a^2\ b^2\ c^2$,

and reads it, two a square, b square, c square plus three a square, b square, c square. The class responds instantly, $5a$ square, b square, c square).

T. But perhaps I ought not to have given you that. (Erases it.) What is the fundamental principle in all this — the fundamental principle in addition?

A. Only like things can be added.

T. Right. It makes no difference what they are, if they are alike they can be added. (Points to the $5\ a$ in the example $2\ a$)
$+3\ a$

$5\ a$. What does this $5a$ mean? You told me that the 5 meant something, and that the a meant something. What does the combination mean? We may understand each of two things without understanding the combination, because the combination may be quite different from either of the things — think of a match and some gunpowder, for instance. How is it here?

A. The 5 means five units, and the a stands for something.

T. What does the a stand for?

A. For anything.

T. $5a$ stands for five anything, then. What does this mean? (Pause.)

T. (Referring back to the $5c = 1c + 1c + 1c + 1c + 1c$). Compare it with this.

Pupil writes $5a = 1a + 1a + 1a + 1a + 1a$.

T. Are we now ready for the original question ? John has a cents and James has a cents ; how many have both ?

A. $2a$ cents.

(Teacher writes $a + a = 2a$.)

T. I wrote $a + a = 2a$; is that right, or ought I to have written $1a + 1a = 2a$?

A. It is right as it is.

T. Why?

A. That 1 is understood, because there is only the a there.

T. I will rub out the 1's and leave only the a's here. (Rubs out the 1's, leaving $5a = a + a + a + a + a$.) Now you are ready for the next point. If I express the operation, what sign shall I put between the 5 and the a?

A. Plus.

T. (Writing $5a = 5 + a$). Is that right ?

A. No.

Pupil. Five equals a.

T. (Writes $5 = a$). You do not mean that, I am sure.

Pupil. It's multiplication, five times a.

T. That is right. If we take a and use it five times that is multiplication. (Puts a period between the 5 and the a.) Instead of the $\times$ we use a dot in algebra to indicate multiplication, because we want to use the $\times$ for another purpose. Now, in ab what is the relation between the a and the b?

A. a times b.

T. Is the relation the same in "arithmetic"? $ab = a \cdot b$; does $25 = 2 \times 5$?

Pupil. No, I do not think it is the same.

T. What is the relation between the 2 and the 5 in 25 ?

(After a number of mistaken answers, one pupil said :

It is addition, bearing in mind the fact of place; that is, two tens or twenty, plus five).

T. (Writes $25 = 20 + 5$.) Now, who is ready to state this third difference ?

A. 3. The relation between digits in a number is that of addition bearing in mind the fact of place ; while the relation between the letters in algebra is that of multiplication.

T. For tomorrow you may try to apply some of these things. I will start you. (Writes on the board $\left.\begin{matrix} 2 \times 2 = 4 \\ 2 + 2 = 4 \end{matrix}\right\}$ Therefore the sum of two numbers equals their product.) Who objects ?

Pupil. It's true there, but not usually.

T. (Writes $\frac{20}{21} \div \frac{4}{7} = \frac{20 \div 4}{21 \div 7} = \frac{5}{3}$). Therefore to divide by a fraction one

may divide the numerator of the dividend by the numerator of the divisor to get the numerator of the quotient; and the denominator of the dividend by the denominator of the divisor to get the denominator of the quotient. Why not? (This had been discussed fully in arithmetic.)

A. You can't always do it.

T. Now it is possible that our proofs in multiplication and division oı frctions may be only special cases like the ones I have given you. How will it be if we put letters in the place of the figures, $\frac{a}{b} \cdot \frac{c}{d}$ instead of $\frac{4}{5} \times \frac{2}{3}$, for instance?

A. That will make the proof more general.

T. For the next time, then, you may substitute letters for figures in your proofs of multiplication and division of fractions.

Notebooks were not used because the class cannot do this kind of thinking and take notes at the same time. Several days later this lesson was reviewed with especial reference to note-taking. Pupils keep two kinds of notebooks, temporary and permanent—*daybooks* and *ledgers*. Before the notes are put into the permanent books abundant opportunity for correction is given, and so the material in the daybooks is being continually revised.

If any pupil has a statement that he prefers to the one agreed upon by the rest, he is allowed to use it if it is a good one; usually, however, there is only one really good statement of a mathematical principle.

A subsequent lesson, No. II, began, after a review by showing the difference between $a+a+a+a+a$ and $a \cdot a \cdot a \cdot a \cdot a$. The outcome of the discussion was to formulate the difference between $5a$ and a^5. Some of the children supplied the term "exponent;" the teacher gave them the word "coefficient."

The lesson then proceeded:

T. John, you may define "coefficient."

(John cannot; no one volunteers.)

T. I see we must try again. Once more, what do we call the a's in the scond case?

A. Factors.

T. And the five?

A. An exponent.

T. Make a definition of exponent.

A. An exponent indicates a certain number of factors.

T. (Writes it.) Very well, and a coefficient indicates a certain number of ——. (No response.) What do we call the *a*'s in the first case?

A. Addends — A coefficient indicates a certain number of addends.

T. That is partly right; we will leave it for a time. I do not know just how to get at the thing I want next, but I will try this way: One boy has five cents; another boy has three cents; what is their combined capital?

A. Eight cents.

T. One boy has eight cents in his pocket, and another boy is five cents in debt; they form a partnership just as the first two did; what is their combined capital?

A. Three cents.

T. In the first, how is the result obtained?

A. By addition.

T. In the second?

A. By subtraction.

T. How many agree? (Twenty-two agreed; four disagreed.) Indicate on the board the first operation. Pupil writes

$$\begin{array}{r} 5\,c \\ +3\,c \\ \hline 8\,c \end{array}$$

T. Now the other. (Pupil writes

$$\begin{array}{r} 8\,c \\ 5\,c \\ \hline 3\,c) \end{array}$$

T. What sign would you put before the five?

A. Minus. (Pupil writes it.)

T. (To pupil who voted in the minority.) What do you mean by saying that the second case is no different from the first?

B. The second boy did not have any money, so he had to take the money of the first by adding,

T. I do not quite see that. (To another of the four.) What is your reason?

Ellis. The proof of subtraction is addition.

T. I do not see that either. (To a third one of the four.) What is your reason?

A. Five cents plus how many cents equal eight cents.

T. That is subtraction — the computer's method, the same as Ellis' idea. (To last of the four.) What do you say?

A. I do not see how it can be anything but addition, because it is the combined capital that is three cents.

T. That is right; go back to your definition of addition, what is that?

A. Addition is the process of combining quantities.

T. The very fact that I said what is their *combined* capital make this addition, by definition. It is in a large sense, addition; in another sense, the majority is right; the operation itself is what you have always called sub-

traction. This brings us to another difference, a fourth one, between "arithmetic" and "algebra." Let us illustrate: A boy has ten cents, his uncle gives him five cents, and he spends a quarter; what is his financial condition?

A. (Given at once.) Ten cents in debt.

T. Indicate this on the board.

(Pupil writes
$$\begin{array}{l} 25\,c \\ 15\,c \\ \hline 10\,c \text{ in debt.)} \end{array}$$

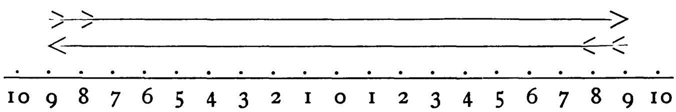
T. Write the whole example and put it in better order.

(Pupil writes
$$\begin{array}{r} 10 \\ +\ 5 \\ \hline 15 \\ -25 \\ \hline 10\,c \text{ in debt.)} \end{array}$$

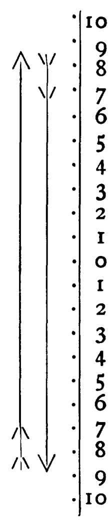
T. How else may you indicate that this is ten cents in debt rather than ten cents in hand?

(Pupil erases the "in debt" and puts a minus sign before the 10.)

T. We will now go at this in another way. (Teacher draws scale.)



We will try an experiment here. Put your index fingers at about the middle of the lower edge of your desks. Those who are right handed use your right hands and those who are left handed, your left. Now move your fingers an inch or so at a time as I point and count at the board; to the left first. (Teacher starts at zero, points and counts to the left, 1, 2, 3, 4, 5, 6, 7, 8, and pupils moves hands as stated.) Now put your fingers back at the starting point and move to the right in the same way. (This was done.) Now I am going to mark one of the directions indicated by these arrows, positive and the other negative. How many here have a decided opinion as to which direction should be marked positive and which negative? (There was no response. This is unusual in my experience.) I will ask the same thing in another way. (Teacher draws a vertical scale to 10.) Of course you immediately think of a thermometer. Are there any who had no preference before, who have a choice on this scale?



Pupil. I do not see what you mean.

T. We have two kinds of quantities in algebra, positive and negative. To illustrate this I have drawn these scales. We are to call one direction positive and the opposite negative. On this thermometer scale, for instance, if we call distance down positive, we must call distance up negative and *vice versa;* on the other, too, distance in one direction will be called positive and in the opposite direction, negative. Now once more try the experiment of moving your hands. This time go clear across with one motion instead of a little at a time, as before. (This was done.) Now how many have a preference with regard to the horizontal scale? (Twenty had a preference; six had none.) How is it with regard to the vertical scale? (All but three had a preference here. One of the pupils who had a choice with regard to the second scale, but not with regard to the first, was called to the board.)

T. Locate upon the scale positive 5 and negative 5. (Pupil marked the 5 above plus and that below minus.)

T. Now, quickly, before you have time to think too much, do the same thing on the other scale. (He marked the 5 at the right minus and the 1 at the left plus.)

T. How many agree with the first, positive up and negative down? (Twenty-three agreed.)

T. With the second, negative to the right and positive to the left? (Six agreed, nineteen disagreed, one had no choice.) (The signs of these 5's were now changed and the sign + was put with the arrows pointing up and right, and the sign – with the others.)

T. Let us now make a list of things we shall call positive and of those we shall call negative. I will write for you. (Teacher writes:

+	–
Positive	Negative)

We have already had money in hand (cash) and debts; how do these go?

A. Cash, positive; debts, negative.

T. The better terms are probably assests and liabilities. (Writes.) (Pupils add above zero, positive; below zero, negative; so above and below normal, right and left hands.)

T. Walter, you may look up for us next time the origin and the meaning of the words *dexterous* and *sinister*, and see if this has anything to do with the matter we have been talking about, and you may all continue this list just as far as you can. By the way, who of you are left-handed, and how did you vote?

(Two were left-handed; they voted with the majority.)

T. Now take this question: Add positive 5 and negative 3. What will that mean on the scale? Herbert, take the pointer and show us.

(H. starts at +5 and goes to –3.)

T. That is what we call subtraction in algebra — finding how many units lie between two numbers. Another answer?

A. Travel from positive 5 three places to the left.

T. Right, and this lands you where?

A. At positive 2.

T. Write it.

(Pupil wrote 2 and then put + before it.)

T. Always write first the sign which shows the direction, and after that the number, which shows where you arrive. There is one other point before we leave this. You all said a little while ago that we were to start at positive 5. Is there any other opinion about this? If we go back further, where shall we begin? Where did this +5 come from? Did anybody have it to start with? Where must everyone begin?

A. At zero.

T. (Diagrams below the scale 0 ______________________ +5 and says):

+2

The whole operation then is from zero five units to the right and then back three units to the left, stopping at positive 2. (This prepares the way for the idea that the first term in a series may be thought of as added to or subtracted from zero, and also for some important work in physics later.)

T. Your next lesson will be: (1) Complete your list of positives and negatives. (2) Work a number of examples, using a scale, and decide about the nature of the sign in the result. Who sees from the examples already given several heads under which all such examples will fall?

A. Three: both positive, both negative, one positive and the other negative.

T. (Continuing.) (3) Complete the sentence: "A coefficient indicates a certain number of ——."

REMARKS

The outcome of this lesson may be of interest. In answer to the first question pupils brought in long lists of positives and negatives; these were written upon the board, and the pupils divided them into two classes: those about which there could never be, as they said, any difference of opinion, and those which might be called positive in some cases and negative in others. It is interesting to note that they were unanimous in calling good positive, and bad negative, right positive and wrong negative, but insisted that these should not be put into the same class with pleasant and unpleasant, beautiful and ugly, satisfactory and unsatisfactory, etc. (Children seem to be usually transcendental in their ethics.)

They were asked whether the photographic negative, and the positive and negative plates of an electric cell are named as they

would expect them to be. The first question was answered affirmatively at once. Most of the class knew nothing about electric cells, and so could not answer the second question. A pupil who did know was therefore called upon to give what he considered the necessary information. He said that the acid worked upon the zinc but not on the carbon. The class said immediately that the zinc plate ought to be postive and the carbon plate negative, "Because," as one put it, "there is something going on at the zinc plate, while there isn't at the other."

The law of signs was correctly stated by several pupils. (Those who do not succeed in discovering things for themselves must, of course, get them as they are given by others in class). It appeared, however, from a question asked at this point, that not all had understood why the operation is really addition in the case where the signs are different. This was therefore discussed at length, the illustrations being diagrams showing applications of the law of physics: "Every force has its full and due effect, whether it acts alone or in connection with other forces." *E. g.*: Problem—One force alone would move a weight from a certain place, which we will call zero, 5 units to the right in a second; another force would move it 3 units to the right in the same time. What effect would both forces have (*a*) if they worked separately each for one second? (*b*) if they worked together for one second? They said that if the forces worked separately one would move the weight 5 units to the right, and then the other would move it 3 units farther, so that it would be at $+8$ at the end of one second. There was some difficulty about case (*b*), but they said finally that if the law was true, the weight must go to $+8$ just the same, only it would get there now in one second instead of two. They all saw that this is addition. Then they were asked to solve the above problem when the second force acts in a direction opposite to that of the other. They said that in case (*a*) the weight would move 5 units to the right in the first second, and 3 units to the left (back 3 units) in the second second; in case (*b*) they finally decided that the weight would move to the right, arriving at $+2$ at the end of one second, and that this, too, must be addition, because the

2 represented the combined effect of the two forces just as much as the 8 in the first illustration did.

The statement in the third question was completed by adding the words *addends* or *subtrahends*. The teacher gave the word *terms* to include both of these.[1] A list of expressions like "$5," "5 bushels," etc., was written upon the board, and the pupils readily completed the sentence, "A term is an expression that represents a certain ——," by inserting "value, "amount," or "quantity." Then the signs + and – were placed before the numbers previously written and reference was made to the lists of positives and negatives upon the board; the pupils were asked why they classified them as they did, and finally worked out the second part of the definition: "and a certain tendency, direction, or *quality*. If this tendency is in a direction that seems naturally satisfactory, desirable, or progressive, the term is called 'positive;' if in the opposite direction it is called 'negative.'"[2]

There are just three ideas in beginning algebra: terms, factors, and equations. Between the first two exists a one to one correspondence that clears up the thinking in elementary algebra wonderfully when it is made explicit. The following is a partial table:

TERMS.	FACTORS.
1. Terms are expressions to be added or subtracted.	1. Factors are expressions to be multiplied or divided.
2. A coefficient is a term index.	2. An exponent is a factor index.
3. A positive coefficient indicates a certain number of positive terms (addends in the arithmetical sense).[3]	3. A positive exponent indicates a certain number of direct factors (multipliers).[4]
4. A negative coefficient indicates a certain number of negative terms (subtrahends, as before).[3]	4. A negative exponent indicates a certain number of inverse factors (divisors).[4]

[1] (The pupil's first idea of terms should be that they are the addends and subtrahends of his arithmetic work. This easily develops into the notion of *quality* later. The broader definition of terms was also developed in this lesson.)

[2] This definition was made by Group X in the University of Chicago Elementary School last year. In the present case the teacher gave the outline of the definition, but the pupils furnished all the key words except *quality*.

[3] The first of any series of terms may always be considered as added to or subtracted from zero.

[4] The first of any series of factors may always be considered as multiplying or dividing one.

TERMS.	FACTORS.
5. Similar terms are such as differ not at all or only in their coefficients.	5. Similar factors are such as differ not at all or only in their exponents.
6. Similar terms are added by adding their coefficients.	6. Similar factors are multiplied by adding their exponents.
7. Similar terms are subtracted by subtracting the coefficient of the subtrahend from that of the minuend.	7. Similar factors are divided by subtracting the exponent of the divisor from that of the dividend.
8. A subtrahend may be written as a minuend or *vice versa,* if the sign of its coefficient be changed.	8. A divisor may be written as a multiplier or *vice versa,* if the sign of its exponent be changed.

As to the time when the third idea mentioned, the equation, should be introduced, there is a difference of opinion. The writer would postpone it until considerable other work had been done, until after addition and subtraction at least. The objection to introducing it earlier, is that it gives the pupils the start with the idea that algebra is only a new way of solving examples, while they should think of it first as a new method of abstract thinking. (See introductory paragraphs.)

CLINTON S. OSBORN.

THE ETHICAL CULTURE SCHOOLS,
New York City.